Embeddings: Let K be a number field.

An embedding of K is a non-zero ring homomorphism $K \hookrightarrow \mathbb{C}$

Note: • it must be injective

• we know $\mathbb{Q} \subseteq K$, and if $\sigma : K \hookrightarrow \mathbb{C}$ is an embedding, then

$$\sigma(q) = q \quad \forall \ q \in \mathbb{Q}.$$

$$\sigma(1) = 1 \qquad \sigma(\overbrace{1 + \cdots + 1}^{n}) = \sigma(1) + \cdots \sigma(1)$$
$$= n$$

Fact: if $K = \mathbb{Q}(\alpha)$ then under an embedding α has to go to a conjugate root.

Example: $K = \mathbb{Q}(\sqrt{2})$ here we have 2 embeddings

$$\eta_1 : a + b\sqrt{2} \longmapsto a + b\sqrt{2}$$

$$\eta_2 : a + b\sqrt{2} \longmapsto a + b(-\sqrt{2}) = a - b\sqrt{2}$$

Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{-7})$

$$\sigma_1 : a + b\sqrt{2} + c\sqrt{-7} + d\sqrt{-14} \xmapsto{\text{identity}} a + b\sqrt{2} + c\sqrt{-7} + d\sqrt{-14}$$

$\sigma_2 : a + b\sqrt{2} + c\sqrt{-7} + d\sqrt{-14} \longmapsto a + b(-\sqrt{2}) + c\sqrt{-7} + d(-\sqrt{-14})$

$\sigma_3 : a + b\sqrt{2} + c\sqrt{-7} + d\sqrt{-14} \longmapsto a + b\sqrt{2} + c(-\sqrt{-7}) + d(-\sqrt{-14})$

$\sigma_4 : a + b\sqrt{2} + c\sqrt{-7} + d\sqrt{-14} \longmapsto a + b(-\sqrt{2}) + c(-\sqrt{-7}) + d\sqrt{-14}$

Recall we say an embedding is real if its image lies in $\mathbb{R}$
otherwise we say its complex

So η_1, η_2 are real embeddings

and σ_i are all complex embeddings

σ_1, σ_3 are conjugate since complex conjugation takes σ_1 to σ_3.

Similarly σ_2, σ_4 are conjugate embeddings

if r_1 is the number of real embedding
and r_2 is the number of complex conjugate embedding

Then for $\mathbb{Q}(\sqrt{2})$ we have $r_1 = 2$. $(r_1, r_2) = (2,0)$

for $\mathbb{Q}(\sqrt{2}, \sqrt{-7})$ we have $r_2 = 2$. $(0,2)$

The standard representation:

Def^n: Let K be a number field and $\alpha \in K$

let $A_\alpha : K \longrightarrow K$ given by $x \mapsto \alpha \cdot x$

The association $\alpha \mapsto A_\alpha$ is called the standard representation

Example: $K = \mathbb{Q}(\sqrt{11})$ $\qquad \alpha = 7 + 4\sqrt{11}$

Pick a basis for K given by $\{1, \sqrt{11}\}$

$$A_\alpha : 1 \mapsto \alpha \cdot 1 = 7 + 4\sqrt{11}$$

$$: \sqrt{11} \mapsto \alpha \cdot \sqrt{11} = 44 + 7\sqrt{11}$$

$$A_\alpha = \begin{pmatrix} 7 & 44 \\ 4 & 7 \end{pmatrix} \qquad \left(\text{or } \underline{A_\alpha = \begin{pmatrix} 7 & 4 \\ 44 & 7 \end{pmatrix}} \right)$$

$\beta = \sqrt{11}$ find A_β ?

a $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$\boxed{\text{b} \quad \begin{pmatrix} 0 & 11 \\ 1 & 0 \end{pmatrix}}$

c $\begin{pmatrix} 0 & -11 \\ 1 & 0 \end{pmatrix}$

$\boxed{\text{d} \quad \begin{pmatrix} 0 & 1 \\ 11 & 0 \end{pmatrix}}$

e $\begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}$

A_β: $1 \mapsto 1\sqrt{11} = 0 + 1\sqrt{11}$
$\sqrt{11} \mapsto \sqrt{11}\cdot\sqrt{11} = 11 = 11 + 0\cdot\sqrt{11}$

$$\begin{pmatrix} 0 & 11 \\ 1 & 0 \end{pmatrix}$$

Using prop 1.64 we know

$$A_{\alpha\beta} = A_\alpha \cdot A_\beta \qquad A_{\alpha+\beta} = A_\alpha + A_\beta$$

and $A_{\lambda\alpha} = \lambda A_\alpha$ for $\alpha \in \mathbb{Q}$.

$\gamma = \alpha\cdot\beta = (7+4\sqrt{11})\cdot(\sqrt{11}) = 44 + 7\sqrt{11}$

$$A_\gamma = A_\alpha \cdot A_\beta = \begin{pmatrix} 7 & 44 \\ 4 & 7 \end{pmatrix} \cdot \begin{pmatrix} 0 & 11 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 44 & 77 \\ 7 & 44 \end{pmatrix}$$

$\delta = \alpha + \beta$

$$A_\delta = A_\alpha + A_\beta = \begin{pmatrix} 7 & 44 \\ 4 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 11 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 55 \\ 5 & 7 \end{pmatrix}$$

Using this we see that if $\alpha = a + b\sqrt{11}$

then $A_\alpha = a \cdot I + b A_\beta \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \cdot A_\beta$$

Using this we define for $\alpha \in K$

$$N_{K/\mathbb{Q}}(\alpha) = \det(A_\alpha) \quad =: \underline{\prod_i \sigma_i(\alpha)}$$

$$Tr_{K/\mathbb{Q}}(\alpha) = \text{Trace}(A_\alpha) \quad = \sum_i \sigma_i(\alpha)$$

<u>Prop 1.7.6</u>

where σ_i are the embeddings of $K \hookrightarrow \mathbb{C}$.

$K = \mathbb{Q}(\sqrt{2}, \sqrt{-7})$

$\alpha = \sqrt{2} \quad , \quad \beta = \sqrt{-7}$

Find A_α and A_β ?

Pick basis ~~$\{1, \sqrt{2}, \sqrt{-7}, \sqrt{-14}\}$~~ $\boxed{\{1, \sqrt{-7}, \sqrt{2}, \sqrt{-14}\}}$

$$A_\alpha = A_{\sqrt{2}} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\sqrt{-7} \cdot \sqrt{-7} = \sqrt{49}$

$i\sqrt{7} \cdot -i\sqrt{7} = \sqrt{49}$

$-i\sqrt{7}$

$\sqrt{-7} = i\sqrt{7}$

$\sqrt{-7} \cdot \sqrt{-7} = i^2 \cdot \sqrt{7}^2 = -1 \cdot 7$

$$A_\beta = A_{\sqrt{-7}} = \begin{pmatrix} 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{c} L \\ | \\ \mathbb{Q}(\sqrt{-7}) \\ | \\ \mathbb{Q} \end{array} \qquad \begin{array}{c} L = \mathbb{Q}(\sqrt{2}, \sqrt{-7}) \\ 2\,| \\ K = \mathbb{Q}(\sqrt{2}) \\ 2\,| \\ \mathbb{Q} \end{array}$$

$$N_{K/\mathbb{Q}}(\alpha) = \det \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} = 4 = \prod_i \sigma_i(\sqrt{2})$$

$$= (\sqrt{2})(\sqrt{2}) \cdot (-\sqrt{2})(-\sqrt{2})$$

$$= \{(\sqrt{2})(-\sqrt{2})\}\{(\sqrt{2})(-\sqrt{2})\}$$

$$N_{K/\mathbb{Q}}(\beta) = \det \begin{pmatrix} 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = 49 = \prod_i \sigma_i(\sqrt{-7})$$

$$= (\sqrt{-7})(-\sqrt{-7}) \cdot (\sqrt{-7}) \cdot (-\sqrt{-7})$$

$$= 7 \cdot 7 = 49$$

$$Tr_{K/\mathbb{Q}}(\alpha) = 0$$

$Tr_{K/\mathbb{Q}}(\beta) = 0$

$\gamma = 2\sqrt{2} + \sqrt{-7}$ Find A_γ

and $N_{K/\mathbb{Q}}(\gamma)$, $Tr_{K/\mathbb{Q}}(\gamma)$

$\{1, \sqrt{2}, i\sqrt{7}, i\sqrt{14}\}$

$A_{i\sqrt{7}}$

$1 \longmapsto \sqrt{-7} = \underline{0 + 0} + 1\cdot\sqrt{-7} +$

$\sqrt{2} \longmapsto \sqrt{-14} = \underline{0 \; - -} \quad 1\cdot\sqrt{-14}$

$\sqrt{-7} \longmapsto -7 = -7 + 0 - \cdot$

$\sqrt{-14} = \sqrt{2}\cdot i\sqrt{7} \xmapsto{i\sqrt{7}} \sqrt{2}\, i\sqrt{7}\cdot i\sqrt{7} = \sqrt{2}\, i^2\sqrt{7}^2 = -1\cdot 7\cdot\sqrt{2}$

$= -7\sqrt{2}$

$$\begin{pmatrix} 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$