ALGEBRAIC NUMBER THEORY- REVISION QUIZ

Question 1. (1) Let p, q be prime numbers such that $q^{p-1} \not\equiv 1 \mod p^2$, let α be a root of $x^p - q$ and let $K = \mathbb{Q}(\alpha)$. Show that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

(2) Using this find the ring of integers of $\mathbb{Q}(\sqrt[5]{37})$ and give an example of a ramified prime.

Question 2. Describe the factorization of the following ideals into prime ideals in $\mathbb{Q}(\zeta_{55})$:

$$(13) \qquad (14) \qquad (5)$$

Question 3. Compute the class group of $\mathbb{Q}(\sqrt{168})$.

Question 4. Let K/F be an extension of number fields and assume that $\alpha \in K$ is a root of a monic polynomial with coefficients in $\mathcal{O}_F$. Prove that α is an algebraic integer.

Question 5. (1) Let p be a prime, k a positive integer and ζ_{p^k} be a p^k-th root of unity and let $\lambda_{p^k} = 1 - \zeta_{p^k}$. Show that

$$\mathbb{Z}[\zeta_{p^k}] = \mathbb{Z}[\lambda_{p^k}]$$

and

$$\Delta(\{1, \zeta_{p^k}, \dots, \zeta_{p^k}^{\varphi(p^k)}\}) = \Delta(\{1, \lambda_{p^k}, \dots, \lambda_{p^k}^{\varphi(p^k)}\}).$$

Here φ is the usual Euler totient function.

(2) Show that $\Delta(\{1, \zeta_{p^k}, \dots, \zeta_{p^k}^{\varphi(p^k)}\})$ divides $p^{k\varphi(p^k)}$.

(3) Let p be a prime and $n = p^k$. Let $S = \{1 \leq x \leq n \mid p \nmid x\}$ (i.e the set of elements less than n which are not divisible by p). Show that

$$\prod_{r \in S} (1 - \zeta_{p^k}^r) = p$$

and from this deduce that $\lambda_{p^k}^{\varphi(p^k)}$ divides p in $\mathbb{Z}[\zeta_{p^k}]$.

(4) Using the above prove that if $K = \mathbb{Q}(\zeta_{p^k})$ then $\mathcal{O}_K = \mathbb{Z}[\zeta_{p^k}] = \mathbb{Z}[\lambda_{p^k}]$.