$\boxed{Q\ 2.11}$ Find the following degrees:

(1) $[\mathbb{Q}(\sqrt{5},\sqrt{7}) : \mathbb{Q}]$

(2) $[\mathbb{Q}(\sqrt{5},\sqrt{11}) : \mathbb{Q}(\sqrt{5}+\sqrt{11})]$

(3) $[\mathbb{Q}(\sqrt[4]{7}) : \mathbb{Q}(\sqrt{7})]$

(1) $K = \mathbb{Q}(\sqrt{5},\sqrt{7})$

Use tower law.

$$K \supseteq \mathbb{Q}(\sqrt{5}) \supseteq \mathbb{Q}$$

$$[K:\mathbb{Q}] = \overset{2}{[K:\mathbb{Q}(\sqrt{5})]} \cdot \overset{2}{[\mathbb{Q}(\sqrt{5}):\mathbb{Q}]}$$

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Note that $\sqrt{7} \in K$

and Claim that $\sqrt{7} \notin \mathbb{Q}(\sqrt{5})$

Here the min pol of

$\sqrt{5}$ is x^2-5

(This is since, its monic)

Why? if $\sqrt{7} = a + b\sqrt{5}$

$\Rightarrow 7 = a^2 + 2ab\sqrt{5} + 5b^2$

$\Rightarrow \sqrt{5} \in \mathbb{Q}$ $\times$.

So min. pol of $\sqrt{7}$ over $\mathbb{Q}(\sqrt{5})$ is $x^2 - 7$.

So $[K : \mathbb{Q}(\sqrt{5})] = 2$.

and irreducible by Eisenstein at $p=5$

② $[\mathbb{Q}(\sqrt{5}, \sqrt{11}) : \mathbb{Q}(\sqrt{5} + \sqrt{11})]$

Claim that this deg is 1 i.e

$$\mathbb{Q}(\sqrt{5}, \sqrt{11}) = \mathbb{Q}(\sqrt{5} + \sqrt{11})$$

Clearly $\sqrt{5} + \sqrt{11} \in \mathbb{Q}(\sqrt{5}, \sqrt{11})$

$\Rightarrow \mathbb{Q}(\sqrt{5} + \sqrt{11}) \subseteq \mathbb{Q}(\sqrt{5}, \sqrt{11})$

Claim that $\sqrt{5}, \sqrt{11} \in \mathbb{Q}(\sqrt{5} + \sqrt{11})$ giving the reverse inclusion.

$\alpha = \sqrt{5} + \sqrt{11}$ Note that $\alpha^2 = 16 + 2\sqrt{55}$

$\Rightarrow \sqrt{55} \in \mathbb{Q}(\sqrt{5} + \sqrt{11})$

Note that $\sqrt{5} \cdot (\sqrt{5} + \sqrt{11}) = 5 + \sqrt{55}$

$\sqrt{11} \cdot (\sqrt{5} + \sqrt{11}) = 11 + \sqrt{55}$

$\Rightarrow \sqrt{5} = \dfrac{5 + \sqrt{55}}{\alpha}$ $\sqrt{11} = \dfrac{11 + \sqrt{55}}{\alpha}$

Q 3.3 $K = \mathbb{Q}(\sqrt[5]{7})$ Let $\alpha = 11 + 6(\sqrt[5]{7})^3$

$\gamma = \sqrt[5]{7}$

Find A_α.

Can't just claim that $1, \gamma, \dots, \gamma^4$ is a basis of $K/\mathbb{Q}$.

Need to check this. To do this we note that

min pol of γ is $x^5 - 7$

(Since it's monic and irred by Eisenstein with $p = 7$)

This means $\{1, \gamma, \dots, \gamma^4\}$ is a basis for $K/\mathbb{Q}$ so can compute A_α.

Trick is to note that we know the following

- if $\alpha \in \mathbb{Q}$ then $A_{\alpha\beta} = \alpha \cdot A_\beta \quad \forall \beta \in K$.

- if $\alpha, \beta \in K$ then $A_{\alpha+\beta} = A_\alpha + A_\beta$

First note that $A_{11} = 11 \cdot A_1 = \begin{pmatrix} 11 & & 0 \\ & \ddots & \\ 0 & & 11 \end{pmatrix}$

Secondly Note that

$$A_{\gamma^3} = \begin{array}{c} \begin{pmatrix} 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ \gamma \\ \gamma^2 \\ \gamma^3 \\ \gamma^4 \end{array}$$

So $A_\alpha = A_{11+6\cdot\gamma^3} = A_{11} + A_{6\cdot\gamma^3} = A_{11} + 6 \cdot A_{\gamma^3}$

$$= \begin{pmatrix} 11 & & 0 \\ & \ddots & \\ 0 & & 11 \end{pmatrix} + 6 \cdot \begin{pmatrix} 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Sheet 4 Questions

Let $K = \mathbb{Q}(\alpha)$ $\quad \alpha = \sqrt{5 - 4\sqrt{2}}$

(a) Find min pol of α. $\qquad \pm\sqrt{5 \pm 4\sqrt{2}}$

$$\alpha = \sqrt{5 - 4\sqrt{2}}$$

$$\Rightarrow \alpha^2 = 5 - 4\sqrt{2}$$

$$\Rightarrow (\alpha^2 - 5)^2 = 16 \cdot 2$$

$$\Rightarrow \alpha^4 - 10\alpha^2 - 7 = 0$$

So $x^4 - 10x^2 - 7$ is the min pol.

this is wrong as you Need to prove its irreducible.

To prove its irred we can either

Check if
- has any roots in $\mathbb{Q}$
- is the product of 2 quadratics

Instead we'll prove this by Checking

$[K : \mathbb{Q}] = 4$, since if this is the case

it means $\deg(m_\alpha) = 4$

so since $m_\alpha \mid x^4 - 10x^2 - 7$

$\Rightarrow m_\alpha = x^4 - 10x^2 - 7$ (as they are both monic)

- Since α satisfies a poly of deg 4 $\Rightarrow$

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq 4$$

- Note that $\mathbb{Q}(\sqrt{2}) \subseteq K$

so $[K : \mathbb{Q}] = [K : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$

$\Rightarrow 2 \mid [K : \mathbb{Q}]$

- So need to check K is not $\mathbb{Q}(\sqrt{2})$

but note that $5 - 4\sqrt{2} < 0$ so

$\alpha = \sqrt{5 - 4\sqrt{2}}$ is not real

so $\alpha \notin \mathbb{Q}(\sqrt{2})$. $\Rightarrow K \neq \mathbb{Q}(\sqrt{2})$

$\Rightarrow [K : \mathbb{Q}] = 4$

(2) Find $[K : \mathbb{Q}]$ and $N_{K/\mathbb{Q}}(\alpha)$, $Tr_{K/\mathbb{Q}}(\alpha)$

Note that since $K = \mathbb{Q}(\alpha)$ then

$m_\alpha = C_\alpha$

and $C_\alpha(x) = x^n - Tr_{K/\mathbb{Q}}(\alpha) x^{n-1} + \cdots + (-1)^n N_{K/\mathbb{Q}}(\alpha)$

$\Rightarrow$ Trace is 0 and Norm is -7.

Warning: if $\beta \in K$ then to find the $N_{K/\mathbb{Q}}(\beta)$ and $Tr_{K/\mathbb{Q}}(\beta)$ we need to find the field poly not the min. pol!

e.g. $K = \mathbb{Q}(\sqrt{5})$ and take $1 \in K$

Then the min. pol of 1 is $x-1$

but recall that if $\beta \in K$ then

$$C_\beta = (m_\beta)^{[K:\mathbb{Q}(\beta)]}$$

So $C_1 = (x-1)^2 = x^2 - 2x + 1$

(3) How many real embeddings.

this is the same as number of real roots

$\sqrt[5]{5 \pm 4\sqrt{2}}$ Note that $\sqrt[5]{5+4\sqrt{2}}$

(4) which of the following are alg. integers

$$\frac{\alpha}{3}, \quad \alpha^2 + 5\alpha + 3, \quad \frac{1+\alpha}{2}$$

$\left[\frac{\alpha}{3}\right]$: This has norm $\frac{-7}{3^4}$, so not in $\mathbb{Z}$ $\Rightarrow$ not an alg int.

$\left[\alpha^2+5\alpha+3\right]$ Yes, since its a lin. comb of alg integers

$\left[\frac{1+\alpha}{2}\right]$ One check that this satisfies

$$x^4-2x^3-x^2-3x-1.$$

$\boxed{4.3}$ Show that $\alpha = \frac{\sqrt{1+\sqrt{17}}}{2}$ has integer norm and trace but isn't an alg integer.

You can't just claim $\{1,\alpha,\alpha^2,\alpha^3\}$ is a basis

Best is to compute the min pol.

We can see that α is a root of

$x^4-\frac{1}{2}x^2-1$ So need to check its irred.

Can check that $[\mathbb{Q}(\alpha):\mathbb{Q}]=4$

$\boxed{4.5}$ let K/F an extension of number fields

and $\alpha \in K$ s.t α satisfies a

monic poly with coeffs in $\mathcal{O}_F$ then

α is an alg integer.

let $\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_0 = 0$ $a_i \in \mathcal{O}_F$

Proof: Use theorem 2.1.15 on

$\mathbb{Z}\{\alpha^i a_j : i, j \in \{0, \dots, n\}\}$ $(a_n = 1)$