Motives Study Group
UCL

Study Group Organisers: Alex Betts & Chris Birkbeck

January–March 2018

Organisation

The study groups will meet 12.00–13.30 every Wednesday, beginning on the 10th of January, in room 505 in the mathematics department of UCL. Talks will be 75 minutes long, with a 15-minute question and discussion session at the end. Notes provided by the speakers will be put up on the course website nms.kcl.ac.uk/alexander.betts/motives shortly after each talk.

Speakers will be given some suggested reading materials for their talks, but are also free to chart their own course should they so wish.

Talk titles and briefs

1 Introduction (Alex Betts, 10/1)

In arithmetic and algebraic geometry, one has access to a profusion of different cohomology theories endowed with rich structures such as Galois actions, Hodge filtrations, or isocrystal structures. Due to the existence of various comparison isomorphisms relating these various cohomology theories, Grothendieck posited the existence of a universal cohomology theory, taking values in a conjectural abelian category of motives, through which all the classical cohomology theories should factor. Moreover, this universal cohomology theory is believed to be closely related to L-functions, and makes concrete predictions about how they should behave.

The aim of this first talk is to give an overview of the various standard cohomology theories and comparison isomorphisms between them. It will also touch on the connections with L-functions, to be resumed in a later lecture.

2 Systems of realisations (17/1)

The most natural approach to producing a suitable universal cohomology theory through which the standard cohomology theories factor is just to consider all of these cohomology theories in parallel, in effect declaring a motive to be equal to the system of its realisations.

This talk will build on the introductory talk by surveying Deligne's and Janssen's constructions of a category of motives via their systems of realisations.

Possible references: [Jan95, DMOS82, Del89]

3 Pure motives (Pol van Hoften, 24/1)

In contrast to Deligne's approach starting from the cohomology theories of interest, Grothendieck's proposal for a construction of a category of *pure* motives was to start from geometry, working out a universal way to abelianise the category of smooth projective varieties. The main wrinkle in Grothendieck's construction is the need to choose an equivalence relation on algebraic cycles, for which several candidates exist. This results in several descriptions of the abelian category of pure motives, and their (partial) equivalence is the subject of the standard conjectures.

This talk will survey Grothendieck's construction of a category of pure motives, and the different behaviours one obtains by using different choices of equivalence relations.

Possible references: Scholl's notes on "L-functions and motives" and [Sch94].

4 Applications to L-functions (Chris Birkbeck, 31/1)

One of the main reasons one might care about motives is what they can tell us about L-functions attached to varieties. In this talk we will attach L-functions to motives and show some of their properties. In particular, we will see how one can factorise L-functions of varieties using motives and also discuss the Bloch–Kato conjectures.

Possible references: [Kah15, Kim09] and Bellaiche's "An introduction to Bloch and Katos conjecture".

5 1-motives (7/2)

One peculiar feature of all the standard cohomology theories is that for a smooth projective variety, its cohomology in degrees ≤ 1 agrees with the corresponding cohomology of its Albanese variety. This suggests the rather surprising notion that functor taking a smooth projective variety to its Albanese variety should be viewed as a universal cohomology theory in degrees ≤ 1. By extending this idea to non-smooth non-projective varieties, Deligne developed a theory of 1-motives, built out of the theory of abelian varieties, which forms a universal cohomology theory in degrees ≤ 1, and which has the singular advantage of being completely explicit and amenable to calculations.

This talk will introduce Deligne's category of 1-motives and describe concretely how different realisation functors allow one to recover all of the standard cohomology theories.

Possible references: [BV07, Del74, Jan95].

6 Theory: derived and triangulated categories (Robin Barlett, 14/2)

In preparation for the following talk, this talk will review some of the basic language of derived and triangulated categories, and how this theory provides a powerful refinement of usual cohomology theories.

Possible references: [Kah15].

7 The derived category of motives (21/2)

A major breakthrough in the theory of motives was provided by Voevodsky, who managed to construct a certain triangulated category which satisfies all the properties (realisations etc.) desired of the *derived category* of the supposed abelian category of motives. What is so surprising about Voevodsky's definition is that not only is it unequivocally the "correct" derived category of motives, it also allows one to prove unconditionally many deep results in number theory.

The aim of this talk is to sketch Voevodsky's construction of his category of motives, and to describe some of the simpler realisation functors which allow it to be related to standard cohomology theories.

Possible references: [Lev05].

8 Theory: algebraic K-theory, Milnor K-theory and motivic cohomology (Adam Morgan (?) 28/2)

Inside any conjectural category of motives, one can attempt to study the motive of a variety by studying the group of morphisms into it from some test motive, for example powers of the Tate motive. These particular Hom- and Ext-groups, known as motivic cohomology, should be computed by a cohomology theory for varieties known as algebraic K-theory.

This talk will sketch the basic definitions of algebraic K-theory and Adams operators, as well as making precise the relation with Ext-groups in the category motives.

Possible references: [Mil71].

9 The Milnor conjecture and the (other) Bloch–Kato conjecture (Carl Wang-Erickson, 7/3)

The Milnor and Bloch–Kato conjectures provide a very explicit description of certain Galois cohomology groups of powers of μ_p in terms of the original field, generalising the classical result that $\mathrm{H}^1(G_F, \mu_p) \cong F^\times/F^{\times p}$. Surprisingly, although these results are purely statements in Galois cohomology, the proof requires using motivic cohomology and the tools developed by Voevodsky.

This talk will introduce these two fundamental conjectures and discuss in outline the methods that are used to prove them.

Possible references: [MVW06, Kah05], D. Dugger's "Notes on the Milnor conjectures".

10 Mixed Tate motives (Netan Dogra (?) 14/3)

Throughout the preceding talks, a special role has been played by the Tate motive, which describes the homology of the torus $\mathbb{G}_m$, and which is closely tied to special values of zeta and multiple zeta functions. One of the major applications of the conjectures that can be proved in Voevodsky's category is that one can unconditionally describe a t-structure on the subcategory generated by powers of the Tate motive, which provides an unconditional construction of an abelian category of mixed Tate motives.

This talk will describe Levine's construction of the category of mixed Tate motives out of Voevodsky's category.

Possible references: [Lev05].

References

[BV07] Luca Barbieri-Viale. On the theory of 1-motives. In *Algebraic cycles and motives. Vol. 1*, volume 343 of *London Math. Soc. Lecture Note Ser.*, pages 55–101. Cambridge Univ. Press, Cambridge, 2007.

[Del74] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

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[Kah05] Bruno Kahn. La conjecture de Milnor (d'après V. Voevodsky) [mr1627119]. In *Handbook of K-theory. Vol. 1, 2*, pages 1105–1149. Springer, Berlin, 2005.

[Kah15] B. Kahn. Fonctions z\ˆeta et L de vari\'et\'es et de motifs. *ArXiv e-prints*, December 2015.

[Kim09] Minhyong Kim. Classical motives and motivic L-functions. In *Autour des motifs—École d'été Franco-Asiatique de Géométrie Algébrique et de Théorie des Nombres/Asian-French Summer School on Algebraic Geometry and Number Theory. Volume I*, volume 29 of *Panor. Synthèses*, pages 1–25. Soc. Math. France, Paris, 2009.

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[Sch94] A. J. Scholl. Classical motives. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 163–187. Amer. Math. Soc., Providence, RI, 1994.