ALGEBRAIC NUMBER THEORY- SHEET 2

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For this problem sheet you have a choice:

- Option 1: Hand in solutions to 2.5 and 2.14
- Option 2: Hand in a solution to 2.13

Solutions should be uploaded to moodle by: 11:59pm on 31/01/2021
[The $*$'s denote hard questions.]

Exercise 2.1. Let F be a field. Check that the only units in $F[x]$ are given by polynomials of degree 0, i.e., they are elements of F.

Exercise 2.2. True or False: If $p(x)$ is reducible in $F[x]$ then there exists $\alpha \in F$ such that $f(\alpha) = 0$ (i.e it has a root in F).

Exercise 2.3. Show that $x^3 + 8x^2 + 3x + 1$ is irreducible in $\mathbb{Q}[x]$.

Exercise 2.4. Prove that $f(x) \in \mathbb{Z}[x]$ is irreducible if and only if $f(x + a)$ is irreducible for any $a \in \mathbb{Z}$.

Exercise 2.5. Prove that $p(x) = x^2 + x + 1$ is irreducible in $\mathbb{F}_2[x]$.

Exercise 2.6. Calculate the minimal polynomials over $\mathbb{Q}$ of the following numbers and show that the polynomials are irreducible.

(1) $2\sqrt[3]{6} + 3$
(2) $\sqrt{3} + \sqrt{5}$
(3) $e^{2\pi i/12}$.
(4) $\sqrt{7 + \sqrt{3}}$

Exercise 2.7. Let L/F be a field extension and let α be algebraic over F, then prove that $m_{\alpha,L}(x)$ divides $m_{\alpha,F}(x)$ in $L[x]$.

Exercise 2.8. Prove that if $\alpha \in K$ is algebraic over F, then it is also algebraic over any field extension of F.

Exercise* 2.9. Show that $x^4 + 1$ is irreducible in $\mathbb{Q}[x]$ but factorizes modulo every prime number p.

Exercise 2.10. Let $\pi = 3.14\ldots$ and $e = 2.71\ldots$ be the usual transcendental numbers. Show that one of $\pi + e$ and πe must also be transcendental.

Exercise 2.11. Calculate the following degrees:

(1) $[\mathbb{Q}(\sqrt{5}, \sqrt{7})/\mathbb{Q}]$
(2) $[\mathbb{Q}(\sqrt{5}, \sqrt{11})/\mathbb{Q}(\sqrt{5} + \sqrt{11})]$
(3) $[\mathbb{Q}(\sqrt[4]{7})/\mathbb{Q}(\sqrt{7})]$

Exercise 2.12. Let p, q be odd prime numbers, with $p \neq q$. Show that

$$\mathbb{Q}(\sqrt{p} + \sqrt{q}) = \mathbb{Q}(\sqrt{p}, \sqrt{q}).$$

Exercise* 2.13. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots \pm p$ with $a_i \in \mathbb{Z}$ and p a prime number. Prove that if

$$p > 1 + |a_1| + \cdots + |a_{n-1}|$$

then f is irreducible in $\mathbb{Z}[x]$.

Exercise 2.14. Let α be an algebraic number such that m_α has degree n. Let $f, g \in \mathbb{Z}[x]$ be of degree strictly less than n such that $f(\alpha) = g(\alpha)$. Show that $f = g$.